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On the confinement of a Dirac particle to a two-dimensional ring

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ABSTRACT

In this contribution, we propose a new model for studying the confinement of a spin-half particle to a two-dimensional quantum ring for systems described by the Dirac equation by introducing a new coupling into the Dirac equation. We show that the introduction of this new coupling into the Dirac equation yields a generalization of the two-dimensional quantum ring model proposed by Tan and Inkson [W.-C. Tan, J.C. Inkson, *Semicond. Sci. Technol.* 11 (1996) 1635] for relativistic spin-half quantum particles.

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1. Introduction

Recent studies of the interaction between a relativistic spin-half particle with the harmonic oscillator potential has shown, in the nonrelativistic limit of the Dirac equation, the impossibility of recovering the harmonic oscillator Hamiltonian due to the presence of a quadratic potential [1–4]. Therefore, the relativistic harmonic oscillator has been investigated by introducing a coupling into the Dirac equation in such a way that the Dirac equation remains linear both in momenta and coordinates, and such that one can recover the Schrödinger equation for a harmonic oscillator in the nonrelativistic limit of the Dirac equation. This new coupling introduced into the Dirac equation is called the Dirac oscillator [1]. Another way of obtaining a relativistic oscillator was investigated in [5] by introducing a scalar and vector potentials which are quadratics in coordinates. In recent years, the Dirac oscillator has been used in studies of the Ramsey-interferometry effect [4], in the quantum Hall effect [6], and has also been investigated in the presence of an external magnetic field [7].

It is worth mentioning that the introduction of the Dirac oscillator does not allow us to make a complete study of the confinement of a relativistic quantum particle to a two-dimensional quantum ring. In the nonrelativistic context of the quantum mechanics, the confinement of quantum particles into a two-dimensional quantum ring has presented interesting results, such as, a non-

parabolic spectrum of energy [8], the arising of persistent currents due to the dependency of the energy levels on the Berry's phase [9], a parabolic spectrum of energy [10–13], and the arising of persistent currents due to the dependency of the energy levels on the Aharonov–Casher quantum flux [13,14].

In this Letter, based on the coupling corresponding to the Dirac oscillator [1], we propose the introduction of a new coupling into the Dirac equation to study the confinement of a spin-half particle to a two-dimensional quantum ring in condensed matter systems described by the Dirac equation. This is the case of an ultra-relativistic quasiparticle system, where the system is characterized by a linear dispersion relation with respect to the velocity obeyed by the quasiparticles. For instance, near the Fermi points in graphene, the dispersion relation is linear with respect to momenta, where the Fermi velocity plays the role of velocity of the light [15,16]. We show that the introduction of this new coupling into the Dirac equation yields a generalization of the two-dimensional quantum ring model proposed by Tan and Inkson [8] for relativistic quantum particles. By using this model, we can also obtain a nonparabolic and discrete spectrum of energy for a spin-half particle confined to two-dimensional quantum ring analogous to the Tan–Inkson model [8] in the nonrelativistic limit. Furthermore, we show that this relativistic model allows us to discuss both the confinement of a spin-half particle to a quantum dot, and the interaction with a quantum antidot in systems described by the Dirac equation.

The structure of this Letter is as follows: in Section 2, we begin by introducing a new coupling into the Dirac equation to describe the confinement of a relativistic spin-half particle to a two-dimensional quantum ring; in Section 3, we consider the

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presence of a magnetic flux in the center of the two-dimensional quantum ring, and solve the Dirac equation exactly. We show that the relativistic energy levels has a dependence of the magnetic quantum flux which gives rise to the arising of persistent currents in the two-dimensional quantum ring; in Section 4, we discuss the nonrelativistic limit of the Dirac equation; in Section 5, we discuss the limit where this model describes the confinement of a spin-half particle to a quantum dot, and the interaction with a quantum antidot in systems described by the Dirac equation. In Section 6, we present our conclusions.

2. Two-dimensional quantum ring model

Let us begin by introducing a new coupling into the Dirac equation. As we have discussed in the introduction, the interaction between a relativistic spin-half particle with the harmonic oscillator potential has shown problems in recovering the harmonic oscillator Hamiltonian in the nonrelativistic limit of the Dirac equation due to the presence of a quadratic potential [1–4]. Therefore, a new coupling has been introduced into the Dirac equation in such a way that this equation remains a linear equation both in momenta and coordinates in order to study the relativistic harmonic oscillator. This coupling, which is known in the literature as the Dirac oscillator [1], is given by

$$\vec{p} \rightarrow \vec{p} - im\omega\rho\gamma^0\hat{\rho}, \quad (1)$$

where $\hat{\rho}$ is the unit vector in the radial direction. In this way, based on the coupling which gives rise to the Dirac oscillator, we introduce the following coupling into the Dirac equation

$$\vec{p} \rightarrow \vec{p} + i\left[\frac{\sqrt{2ma_1}}{\rho} + \sqrt{2ma_2}\rho\right]\gamma^0\hat{\rho}, \quad (2)$$

where a_1 and a_2 are control parameters. In the following, we show that the introduction of the coupling (2) gives rise to a model for studies of the confinement of a relativistic spin-half particle to a two-dimensional quantum ring yielding a generalization of the Tan-Inkson model for a two-dimensional quantum ring [8]. We also show that, by taking the parameter $a_1 = 0$, the relativistic energy levels correspond to both the energy levels of the Dirac oscillator, and the confinement of a relativistic spin-half particle to a quantum dot. Moreover, by taking the parameter $a_2 = 0$, we have the case where the relativistic spin-half particle interacts with a quantum antidot, and no bound states can be achieved.

3. Solutions of the Dirac equation in a two-dimensional ring

In this section, we obtain the solutions of the Dirac equation in the presence of the confining potential described by Eq. (2). Since the coupling (2) takes into account the cylindrical symmetry, we work the Dirac equation with curvilinear coordinates. In this case, the line element of the Minkowski spacetime is written in cylindrical coordinates as follows: $ds^2 = -dt^2 + d\rho^2 + \rho^2 d\varphi^2 + dz^2$. Thus, by applying a coordinate transformation $\frac{\partial}{\partial x^\mu} = \frac{\partial \tilde{x}^\nu}{\partial x^\mu} \frac{\partial}{\partial \tilde{x}^\nu}$, and a unitary transformation on the wave function $\psi(x) = U\psi'(\tilde{x})$, the Dirac equation can be written in any orthogonal system in the following form [17]:

$$i\gamma^\mu D_\mu \psi + \frac{i}{2} \sum_{k=1}^3 \gamma^k \left[D_k \ln \left(\frac{h_1 h_2 h_3}{h_k} \right) \right] \psi = m\psi, \quad (3)$$

where $D_\mu = \frac{1}{h_\mu} \partial_\mu$ is the derivative of the corresponding coordinate system, and the parameters h_k are the scale factors of the corresponding coordinate system. For instance, in cylindrical coordinates, the scale factors are given by $h_0 = 1$, $h_1 = 1$, $h_2 = \rho$,

and $h_3 = 1$. In this way, the second term in (3) gives rise to a term called the spinorial connection [17–22]. The matrices γ^μ are the Dirac matrices given in the Minkowski spacetime [23,24], that is,

$$\gamma^0 = \hat{\beta} = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}; \quad \gamma^i = \hat{\beta}\hat{\alpha}^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix};$$

$$\Sigma^i = \begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix}, \quad (4)$$

with I , $\vec{\Sigma}$ and σ^i being the 2×2 identity matrix, the spin vector and the Pauli matrices, respectively. The Pauli matrices satisfy the relation $(\sigma^i \sigma^j + \sigma^j \sigma^i) = 2\eta^{ij}$, where $\eta^{\mu\nu} = \text{diag}(- + + +)$ is the Minkowski tensor.

Let us also consider a field characterized by a potential vector $\vec{A} = \frac{\phi}{2\pi\rho}\hat{\phi}$, with ϕ being the magnetic flux on the z direction, where the magnetic field is given by $\vec{B} = \phi\delta(x)\delta(y)\hat{z}$. The purpose for introducing this magnetic flux or the Aharonov–Bohm flux is to investigate the arising of persistent currents in the nonrelativistic case [8]. In the present case, we have added this flux in order to obtain persistent currents in the quantum ring described by the model (2), and compare the relativistic behavior with the well-known results of the nonrelativistic case [8].

Hence, by introducing the coupling (2) into the Dirac equation in the presence of the magnetic flux, the Dirac equation becomes

$$i\frac{\partial\psi}{\partial t} = m\hat{\beta}\psi - i\hat{\alpha}^1 \left[\frac{\partial}{\partial\rho} + \frac{1}{2\rho} - \hat{\beta}\frac{\sqrt{2ma_1}}{\rho} - \hat{\beta}\sqrt{2ma_2}\rho \right] \psi$$

$$- i\frac{\hat{\alpha}^2}{\rho} \left[\frac{\partial}{\partial\varphi} - i\frac{\phi}{\phi_0} \right] \psi - i\hat{\alpha}^3 \frac{\partial\psi}{\partial z}, \quad (5)$$

where q corresponds to the electric charge of the particle, and $\phi_0 = 2\pi/|q|$. The solution of the Dirac equation (5) is given in the form:

$$\psi = e^{-i\mathcal{E}t} \begin{pmatrix} \eta \\ \chi \end{pmatrix}, \quad (6)$$

where $\eta = \eta(\rho, \varphi, z)$ and $\chi = \chi(\rho, \varphi, z)$ are two-spinors. Thus, substituting (6) into the Dirac equation (5), we obtain two coupled equations for η and χ , where the first coupled equation is

$$(\mathcal{E} - m)\eta = -i\sigma^1 \left[\frac{\partial}{\partial\rho} + \frac{1}{2\rho} + \frac{\sqrt{2ma_1}}{\rho} + \sqrt{2ma_2}\rho \right] \chi$$

$$- i\frac{\sigma^2}{\rho} \left[\frac{\partial}{\partial\varphi} - i\frac{\phi}{\phi_0} \right] \chi - i\sigma^3 \frac{\partial\chi}{\partial z}, \quad (7)$$

while the second coupled equation is

$$(\mathcal{E} + m)\chi = -i\sigma^1 \left[\frac{\partial}{\partial\rho} + \frac{1}{2\rho} - \frac{\sqrt{2ma_1}}{\rho} - \sqrt{2ma_2}\rho \right] \eta$$

$$- i\frac{\sigma^2}{\rho} \left[\frac{\partial}{\partial\varphi} - i\frac{\phi}{\phi_0} \right] \eta - i\sigma^3 \frac{\partial\eta}{\partial z}. \quad (8)$$

By eliminating χ in (8) and substituting it in (7), we obtain the following second-order differential equation:

$$(\mathcal{E}^2 - m^2)\eta = -\frac{\partial^2\eta}{\partial\rho^2} - \frac{1}{\rho}\frac{\partial\eta}{\partial\rho} + \frac{\eta}{4\rho^2} - \frac{\sqrt{2ma_1}}{\rho^2}\eta$$

$$+ \sqrt{2ma_2}\eta + i\frac{\sigma^3}{\rho^2}\frac{\partial\eta}{\partial\varphi} + \frac{2ma_1}{\rho^2}\eta$$

$$+ 4m\sqrt{a_1a_2}\eta - 2i\sigma^3\frac{\sqrt{2ma_1}}{\rho^2}\frac{\partial\eta}{\partial\varphi}$$

$$- 2\sigma^3\frac{\phi}{\phi_0}\frac{\sqrt{2ma_1}}{\rho^2}\eta + 2ma_2\rho^2\eta$$

$$\begin{aligned}
& -2i\sigma^3\sqrt{2ma_2}\frac{\partial\eta}{\partial\phi} - 2\frac{\phi}{\phi_0}\sqrt{2ma_2}\sigma^3\eta \\
& + \frac{\phi}{\phi_0}\frac{\sigma^3}{\rho^2}\eta - \frac{1}{\rho^2}\frac{\partial^2\eta}{\partial\phi^2} \\
& + \left(\frac{\phi}{\phi_0\rho}\right)^2\eta + 2i\frac{\phi}{\phi_0\rho^2}\frac{\partial\eta}{\partial\phi} - \frac{\partial^2\eta}{\partial z^2}, \quad (9)
\end{aligned}$$

where we have assumed that the spin of the quantum particle is polarized along the z -axis. From (9), we can see that η is an eigenfunction of σ^3 , whose eigenvalues are $s = \pm 1$. Thus, we can write $\sigma^3\eta_s = \pm\eta_s = s\eta_s$, that is, $\sigma^3\eta_+ = \eta_+$ and $\sigma^3\eta_- = -\eta_-$. We can also see that the operators $\hat{J}_z = -i\partial_\phi$,¹ and $\hat{p}_z = -i\partial_z$ commute with the Hamiltonian of the right-hand side of (9), then, we can take the solutions of (9) in the form:

$$\eta_s = e^{ij\phi} e^{ikz} \begin{pmatrix} R_+(\rho) \\ R_-(\rho) \end{pmatrix}, \quad (10)$$

where $j = l + \frac{1}{2}$, $l = 0, \pm 1, \pm 2, \dots$, and k is a constant. Note that Eq. (10) is an eigenstate of \hat{J}_z with half-integer eigenvalues given $j = \pm\frac{1}{2}, \pm\frac{3}{2}, \dots$, a fact required for a spin-1/2 particle. In this work, we consider $k = 0$ since we are interested in discussing a planar system. Substituting (10) into (9), we obtain two non-coupled radial equations for $R_+(\rho)$ and $R_-(\rho)$, therefore we write the equations for $R_+(\rho)$ and $R_-(\rho)$ in the following compact form:

$$\left[\frac{d^2}{d\rho^2} + \frac{1}{\rho} \frac{d}{d\rho} - \frac{\tau_s^2}{\rho^2} - 2ma_2\rho^2 + \nu_s \right] R_s(\rho) = 0, \quad (11)$$

where we have defined in Eq. (11), the parameters:

$$\begin{aligned}
\tau_s &= \zeta_s + s\sqrt{2ma_1}; \\
\zeta_s &= l + \frac{1}{2}(1-s) - \frac{\phi}{\phi_0}; \\
\nu_s &= \mathcal{E}^2 + m^2 - 2s\sqrt{2ma_2}\zeta_s - 2\sqrt{2ma_2} - 4m\sqrt{a_1a_2}. \quad (12)
\end{aligned}$$

Note that Eq. (11) has the same form of the radial equation obtained in Ref. [8], that represents a particle in the presence of an effective quantum ring potential. In this way, Eq. (11) describes the motion of the particle confined to a ring geometry.

Before solving the second-order differential equation (11), we make a coordinate transformation given by $\mu = \sqrt{2ma_2}\rho^2$. In this way, we rewrite (11) in the form:

$$\left[\mu \frac{d^2}{d\mu^2} + \frac{d}{d\mu} - \frac{\tau_s^2}{4\mu} - \frac{\mu}{4} + \frac{\nu_s}{4\sqrt{2ma_2}} \right] R_s(\mu) = 0. \quad (13)$$

Thus, in order to have a regular solution at the origin and finite everywhere (that is, $R_s \rightarrow 0$ when $\rho \rightarrow \infty$), we take the solution of (13) in the form:

$$R_s(\mu) = e^{-\frac{\mu}{2}} \mu^{\frac{|\tau_s|}{2}} F_s(\mu). \quad (14)$$

Substituting (14) into (13), we obtain

¹ Note that, in cylindrical coordinates \hat{J}_z is expressed as just a derivative of ϕ . Hence, the solutions of Eq. (9) must be eigenfunctions of $\hat{J}_z = -i\partial_\phi$, whose eigenvalues are $j = \pm\frac{1}{2}, \pm\frac{3}{2}, \dots = l \pm \frac{1}{2}$, where $l = 0, \pm 1, \pm 2, \dots$. Comparing with Ref. [17], we have the situation where the z -component of the total angular momentum is defined as $\hat{J}_z = -i\partial_\phi + \frac{\sigma^3}{2}$ [23,24], but with the γ^μ matrices being defined in terms of the space coordinates, that is, $\bar{\gamma}^0 = \gamma^0$, $\bar{\gamma}^1 = \cos\phi\gamma^1 + \sin\phi\gamma^2$, $\bar{\gamma}^2 = -\sin\phi\gamma^1 + \cos\phi\gamma^2$, and $\bar{\gamma}^3 = \gamma^3$. Hence, the authors of Ref. [17] show that by making a coordinate transformation from the Cartesian coordinates to the cylindrical coordinates, and by applying a unitary transformation on the spinor given by $\psi_c = \hat{S}\psi = \exp[-\frac{i}{2}\phi\sigma^3]\psi$, where ψ_c is the spinor in Cartesian coordinates, we obtain the same solution given in (10).

$$\mu \frac{d^2 F_s}{d\mu^2} + [|\tau_s| + 1 - \mu] \frac{dF_s}{d\mu} + \left[\frac{\nu_s}{4\sqrt{2ma_2}} - \frac{|\tau_s|}{2} - \frac{1}{2} \right] F_s = 0, \quad (15)$$

which corresponds to the confluent hypergeometric equation or the Kummer equation [25]. The regular solution at the origin is called the Kummer function of first kind, which is given by $F_s(\mu) = {}_1F_1\left(\frac{|\tau_s|}{2} + \frac{1}{2} - \frac{\nu_s}{4\sqrt{2ma_2}}, |\tau_s| + 1, \mu\right)$ [25]. Hence, in order to obtain a finite solution everywhere, we must impose the condition where the confluent hypergeometric series becomes a polynomial of degree n [25,26], with $n = 0, 1, 2, \dots$. This occurs when

$$\frac{|\tau_s|}{2} + \frac{1}{2} - \frac{\nu_s}{4\sqrt{2ma_2}} = -n. \quad (16)$$

By taking the parameters defined in (12), then, the condition (16) yields

$$\begin{aligned}
\mathcal{E}_{n,l}^2 &= m^2 + 4\sqrt{2ma_2} \left[n + \frac{l + \frac{1}{2}(1-s) - \frac{\phi}{\phi_0} + s\sqrt{2ma_1}}{2} \right. \\
&\quad \left. + s \frac{(l + \frac{1}{2}(1-s) - \frac{\phi}{\phi_0})}{2} + 1 \right] + 4m\sqrt{a_1a_2}. \quad (17)
\end{aligned}$$

The discrete spectrum of energy (17) corresponds to the energy levels of bound states of a spin-half charged particle confined to a two-dimensional quantum ring in a system described by the Dirac equation in the presence of a magnetic flux. We can see that the energy levels (17) depend on the control parameters a_1 and a_2 , which allow us to compare with condensed matter systems since we can identify the control parameters in such a system. Furthermore, we also observe the flux dependence of the relativistic energy levels (17) with a periodicity $\phi \rightarrow \phi - \phi_0$, thus, we have that $\mathcal{E}_{n,l}(\phi - \phi_0) = \mathcal{E}_{n,l+1}(\phi)$. This flux dependence of the energy levels gives rise to the arising of persistent currents [27] in the two-dimensional quantum ring given by

$$\begin{aligned}
\mathcal{I} &= - \sum_{n,l} \frac{\partial \mathcal{E}_{n,l}}{\partial \phi} \\
&= \frac{q}{2\pi} \sum_{n,l} \sqrt{2ma_2} \left(\frac{\tau_s}{|\tau_s|} + 1 \right) \\
&\quad \times \left\{ m^2 + 4\sqrt{2ma_2} \left[n + \frac{|\tau_s|}{2} + s \frac{\zeta_s}{2} + 1 \right] + 4m\sqrt{a_1a_2} \right\}^{-1/2}. \quad (18)
\end{aligned}$$

We can also observe that the persistent current (18) depends on the control parameters a_1 and a_2 , and the quantum numbers n and l . Comparing with the Tan-Inkson model [8], where the persistent currents do not depend on the quantum number n , we have that the present model allows us to get the information about the quantum number n for persistent currents in condensed matter systems described by the Dirac equation. We also have that the persistent current (18) is a periodic function of the magnetic flux ϕ .

Next, let us obtain the Dirac spinors corresponding to the positive-energy solution of (5). First of all, we should note that, by taking the radial wave functions (14), we can write (10) in the following form:

$$\begin{aligned}
\eta_s &= e^{i(l+\frac{1}{2})\phi} e^{-\sqrt{\frac{ma_2}{2}}\rho^2} (2ma_2)^{\frac{|\tau_s|}{4}} \rho^{|\tau_s|} \\
&\quad \times {}_1F_1(-n, |\tau_s| + 1, \sqrt{2ma_2}\rho^2). \quad (19)
\end{aligned}$$

Thus, in order to obtain the solutions of the Dirac equation (5),

we must solve the coupled equations given in (7) and (8). Substituting (19) into (8), we can obtain the solutions for the two-spinor χ . Hence, the positive-energy solutions of the Dirac equation (5) corresponding to the parallel and anti-parallel components to the z -axis are given by, respectively,

$$\begin{aligned} \psi_+ = & f_{+1} F_1(-n, |\tau_+| + 1, \sqrt{2ma_2}\rho^2) \\ & \times \begin{pmatrix} 1 \\ 0 \\ 0 \\ \frac{i}{(\mathcal{E}+m)} [2\sqrt{2ma_2}\rho + \frac{(\zeta_+ - |\tau_+| + \sqrt{2ma_1})}{\rho}] \end{pmatrix} \\ & + \frac{if_+}{(\mathcal{E}+m)} \left(\frac{2n\sqrt{2ma_2}\rho}{|\tau_+| + 1} \right) \\ & \times {}_1F_1(-n+1, |\tau_+| + 2, \sqrt{2ma_2}\rho^2) \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \end{aligned} \quad (20)$$

and

$$\begin{aligned} \psi_- = & f_{-1} F_1(-n, |\tau_-| + 1, \sqrt{2ma_2}\rho^2) \\ & \times \begin{pmatrix} 0 \\ 1 \\ 0 \\ \frac{i}{(\mathcal{E}+m)} [2\sqrt{2ma_2}\rho - \frac{(\zeta_- + |\tau_-| - \sqrt{2ma_1})}{\rho}] \end{pmatrix} \\ & + \frac{if_-}{(\mathcal{E}+m)} \left(\frac{2n\sqrt{2ma_2}\rho}{|\tau_-| + 1} \right) \\ & \times {}_1F_1(-n+1, |\tau_-| + 2, \sqrt{2ma_2}\rho^2) \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \end{aligned} \quad (21)$$

where we have defined in (20) and (21) the following parameters:

$$\begin{aligned} f_{\pm} = & \left[\frac{\sqrt{8ma_2}\Gamma(n + |\tau_s| + 1)}{\Gamma(n+1)[\Gamma(|\tau_s| + 1)]^2} \right]^{1/2} \\ & \times e^{-i\mathcal{E}t} e^{i(l + \frac{1}{2})\varphi} e^{-\sqrt{\frac{ma_2}{2}}\rho^2} (2ma_2)^{\frac{|\tau_{\pm}|}{4}} \rho^{|\tau_{\pm}|}, \end{aligned} \quad (22)$$

where $\Gamma(x)$ corresponds to the Gamma function. The spinors (20) and (21) correspond to the positive-energy solutions of the Dirac equation (5). Note that the same procedure can be used in order to obtain the negative-energy solutions of the Dirac equation (5).

4. Nonrelativistic limit

Now, let us make a discussion about the nonrelativistic limit of the energy levels (17). We can obtain the nonrelativistic limit of the energy levels (17) by applying the Taylor expansion, up to the first-order terms. Thus, the Taylor expansion yields

$$\begin{aligned} \mathcal{E}_{n,l} \approx & m + \sqrt{\frac{8a_2}{m}} \left[n + \frac{|l + \frac{1}{2}(1-s) - \frac{\phi}{\phi_0} + s\sqrt{2ma_1}|}{2} \right. \\ & \left. + s \frac{(l + \frac{1}{2}(1-s) - \frac{\phi}{\phi_0})}{2} + 1 \right] + 2\sqrt{a_1 a_2}. \end{aligned} \quad (23)$$

Hence, we can see that the nonrelativistic energy levels (23) recover the results of the Tan–Inkson model [8] for a spin-half charged particle confined to a two-dimensional quantum ring, whose frequency depends on the control parameter a_2 , that is, $\omega = \sqrt{\frac{8a_2}{m}}$. We also note the flux dependence of the nonrelativistic

energy levels (23), with a periodicity $\phi \rightarrow \phi - \phi_0$. In this way, we have $\mathcal{E}_{n,l}(\phi - \phi_0) = \mathcal{E}_{n,l+1}(\phi)$. Thus, the persistent currents are:

$$\mathcal{I} = - \sum_{n,l} \frac{\partial \mathcal{E}_{n,l}}{\partial \phi} \approx \frac{q}{4\pi} \sum_{n,l} \sqrt{\frac{8a_2}{m}} \left[\frac{\tau_s}{|\tau_s|} + 1 \right], \quad (24)$$

which can also be obtained from (18) by applying the Taylor expansion, up to the terms of order $\mathcal{O}(m^{-2})$. Note that the persistent current (24) depends on the control parameter a_1 and a_2 , and the quantum number l . This result agrees with the Tan–Inkson model when we consider a spin-half quantum particle. We also observe that the persistent current (24) is a periodic function of the magnetic flux ϕ .

5. Dirac particle confined to a quantum dot

In the following, let us come back to the coupling (2). One should note that by taking the control parameter $a_1 = 0$ in the coupling (2), we recover the Dirac oscillator [1]. For quantum systems in condensed matter described by the Dirac equation, by taking the control parameter $a_1 = 0$, the coupling (2) describes the confinement of a spin-half particle to a quantum dot. Hence, the energy levels corresponding to the confinement of a spin-half particle in a condensed matter system described by the Dirac equation are

$$\begin{aligned} \mathcal{E}_{n,l}^2 = & m^2 + 4\sqrt{2ma_2} \left[n + \frac{|l + \frac{1}{2}(1-s) - \frac{\phi}{\phi_0}|}{2} \right. \\ & \left. + s \frac{(l + \frac{1}{2}(1-s) - \frac{\phi}{\phi_0})}{2} + 1 \right], \end{aligned} \quad (25)$$

where we can also see the flux dependence of the energy levels (25), that is, we also have $\mathcal{E}_{n,l}(\phi - \phi_0) = \mathcal{E}_{n,l+1}(\phi)$. Moreover, the persistent currents inside the quantum dot are

$$\begin{aligned} \mathcal{I} = & - \sum_{n,l} \frac{\partial \mathcal{E}_{n,l}}{\partial \phi} \\ = & \frac{q}{2\pi} \sum_{n,l} \sqrt{2ma_2} \left(\frac{\zeta_s}{|\zeta_s|} + 1 \right) \\ & \times \left\{ m^2 + 4\sqrt{2ma_2} \left[n + \frac{|\zeta_s|}{2} + s \frac{\zeta_s}{2} + 1 \right] \right\}^{-1/2}. \end{aligned} \quad (26)$$

The nonrelativistic limit of the energy levels (25) can also be obtained by applying the Taylor expansion, up to the first-order terms. The corresponding nonrelativistic energy levels are

$$\begin{aligned} \mathcal{E}_{n,l} \approx & m + \sqrt{\frac{8a_2}{m}} \left[n + \frac{|l + \frac{1}{2}(1-s) - \frac{\phi}{\phi_0}|}{2} \right. \\ & \left. + s \frac{(l + \frac{1}{2}(1-s) - \frac{\phi}{\phi_0})}{2} + 1 \right], \end{aligned} \quad (27)$$

which recover the results obtained by Tan and Inkson in [8] for a spin-half particle. We also have a flux dependence in (27) with a periodicity $\mathcal{E}_{n,l}(\phi - \phi_0) = \mathcal{E}_{n,l+1}(\phi)$, therefore the persistent current are [28,29]

$$\mathcal{I} = - \sum_{n,l} \frac{\partial \mathcal{E}_{n,l}}{\partial \phi} \approx \frac{q}{4\pi} \sum_{n,l} \sqrt{\frac{8a_2}{m}} \left[\frac{\zeta_s}{|\zeta_s|} + 1 \right], \quad (28)$$

which can be obtained from (26) by applying the Taylor expansion, up to the terms of order $\mathcal{O}(m^{-2})$.

Furthermore, by taking the control parameter $a_2 = 0$ in (2), we have that no bound states can be achieved anymore. This case

corresponds to the interaction between the spin-half particle and a quantum antidot, where there is no confinement of the quantum particle [8].

6. Conclusions

We have proposed a model for studying the confinement of a spin-half particle to a two-dimensional ring in quantum systems described by the Dirac equation. This model consists in introducing a new coupling into the Dirac equation which depends on two control parameters. In this way, we have shown that we can obtain a generalization of the Tan–Inkson model for a two-dimensional quantum ring [8] for systems described by the Dirac equation, where the energy levels depend on the control parameters. Moreover, we have shown that we can recover the nonparabolic energy levels obtained in the Tan–Inkson model [8] in the nonrelativistic limit of the energy levels. Furthermore, we have considered the presence of a magnetic flux on the z -axis, and shown that the energy levels are flux dependent. Thus, due to the flux dependence of the energy levels, we have discussed the arising of persistent currents in the two-dimensional quantum ring for condensed matter systems described by the Dirac equation.

We have also shown, by taking the control parameter $a_1 = 0$, that we can recover the Dirac oscillator [1] and study the confinement of a spin-half particle to a quantum dot in a condensed matter system described by the Dirac equation. Hence, we have obtained the energy levels, and discussed the arising of persistent currents inside the quantum dot. Moreover, we have discussed the case of the interaction between the spin-half particle with a antidot by taking the control parameter $a_2 = 0$, and shown that no bound states can be achieved. Finally, we have obtained the Dirac spinors corresponding to the positive-energy solutions.

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References

- [1] M. Moshinsky, A. Szczepaniak, *J. Phys. A: Math. Gen.* 22 (1989) L817.
- [2] R.P. Martínez-y-Romero, H.N. Núñez-Yépez, A.L. Salas-Brito, *Eur. J. Phys.* 16 (1995) 135.
- [3] V.M. Villalba, *Phys. Rev. A* 49 (1994) 586.
- [4] A. Bermudez, M.A. Martín-Delgado, A. Luis, *Phys. Rev. A* 77 (2008) 033832.
- [5] A.S. de Castro, P. Alberto, R. Lisboa, M. Malheiro, *Phys. Rev. C* 73 (2006) 054309.
- [6] N. Ferkous, A. Bounames, *Phys. Lett. A* 325 (2004) 21.
- [7] B.P. Mandal, S. Verma, *Phys. Lett. A* 374 (2010) 1021.
- [8] W.-C. Tan, J.C. Inkson, *Semicond. Sci. Technol.* 11 (1996) 1635; W.-C. Tan, J.C. Inkson, *Phys. Rev. B* 53 (1996) 6947; W.-C. Tan, J.C. Inkson, *Phys. Rev. B* 60 (1999) 5626.
- [9] D. Loss, P. Goldbart, *Phys. Rev. B* 45 (1990) 13544; D. Loss, P. Goldbart, A.V. Balatsky, *Phys. Rev. Lett.* 65 (1990) 1655; A.G. Aronov, Y.B. Lyanda-Geller, *Phys. Rev. Lett.* 70 (1993) 343.
- [10] N. Aquino, E. Castaño, E. Ley-Koo, *Chinese J. Phys.* 41 (2003) 276.
- [11] L. Solimany, B. Kramer, *Solid State Commun.* 96 (1995) 471.
- [12] A.L. Silva Netto, C. Chesman, C. Furtado, *Phys. Lett. A* 372 (2008) 3894.
- [13] K. Bakke, C. Furtado, *Mod. Phys. Lett. A* 26 (2011) 1331.
- [14] S. Oh, C.-M. Ryu, *Phys. Rev. B* 51 (1995) 13441; H. Mathur, A.D. Stone, *Phys. Rev. B* 44 (1991) 10957; H. Mathur, A.D. Stone, *Phys. Rev. Lett.* 68 (1992) 2964; A.V. Balatsky, B.L. Altshuler, *Phys. Rev. Lett.* 70 (1993) 1678.
- [15] P.E. Lammert, V.H. Crespi, *Phys. Rev. Lett.* 85 (2000) 5190.
- [16] P.E. Lammert, V.H. Crespi, *Phys. Rev. B* 69 (2004) 035406.
- [17] P. Schluter, K.H. Wietschorke, W. Greiner, *J. Phys. A* 16 (1983) 1999.
- [18] K. Bakke, C. Furtado, *Phys. Rev. A* 80 (2009) 032106.
- [19] S. Weinberg, *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity*, IE-Wiley, New York, 1972.
- [20] K. Bakke, *Phys. Lett. A* 374 (2010) 4642; K. Bakke, *Ann. Phys. (Berlin)* 523 (2011) 762.
- [21] K. Bakke, H. Belich, E.O. Silva, *J. Math. Phys.* 52 (2011) 063505.
- [22] K. Bakke, H. Belich, E.O. Silva, *Ann. Phys. (Berlin)* 523 (2011) 910.
- [23] J.M. Bjorken, S.D. Drell, *Relativistic Quantum Mechanics*, McGraw-Hill Book Company, New York, 1964.
- [24] W. Greiner, *Relativistic Quantum Mechanics: Wave Equations*, 3rd ed., Springer, Berlin, 2000.
- [25] M. Abramowitz, I.A. Stegun, *Handbook of Mathematical Functions*, Dover Publications Inc., New York, 1965.
- [26] L.D. Landau, E.M. Lifshitz, *Quantum Mechanics, The Nonrelativistic Theory*, 3rd ed., Pergamon, Oxford, 1977.
- [27] N. Byers, N.C. Yang, *Phys. Rev. Lett.* 7 (1961) 46.
- [28] Y. Avishai, M. Kohmoto, *Phys. Rev. Lett.* 71 (1993) 279.
- [29] C. Furtado, A. Rosas, S. Azevedo, *Europhys. Lett.* 79 (2007) 57001.